Unification of Gravitation and Gauge Fields

Xin-Bing Huang*

Department of Physics, Peking University,

100871 Beijing, China

Abstract

In this letter, I indicate that complex daor field should also have spinor suffixes. The gravitation and gauge fields are unified under the framework of daor field. I acquire the elegant coupling equation of gravitation and gauge fields, from which Einstein's gravitational equation can be deduced.

PACS numbers: 04.20.Cv, 04.20.Gz, 03.65.Vf, 12.10.Kt

*huangxb@mail.phy.pku.edu.cn

About four decade years ago, some physicists recognized the fact that Yang-Mills gauge theories and the affine geometry of principal fiber bundles are one[1, 2]. But Einstein's gravitational theory is the affine geometry of tangent bundles. They seems to be quite different. I have indicated that daor field will construct a possible connection between them[3]. This letter is devoted to this topic: Gravitation and gauge field are unified in a harmonic structure, and the coupling equation is set up, which is consistent with Einstein's gravitational theory.

Suppose an ideal universe, in which there is no matter present except the gravitational field and gauge fields. Einstein's gravitational equation can be written as[†][4, 5]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ , \tag{1}$$

where G is Newtonian gravitational constant, and $T_{\mu\nu}$ is the stress-energy tensor for gauge fields. In the case of electromagnetic field, $T_{\mu\nu}$ is given by

$$4\pi T_{\mu\nu} = \mathbf{f}_{\mu}{}^{\alpha}\mathbf{f}_{\nu\alpha} - \frac{1}{4}\, g_{\mu\nu}\mathbf{f}_{\alpha\beta}\mathbf{f}^{\alpha\beta} \ , \tag{2}$$

where $\mathbf{f}_{\alpha\beta}$ is the strength of electromagnetic field. I adopt the same sign conventions as in Misner-Thorne-Wheeler's book[4]. The metric tensor of Minkowski space-time η_{ab} is written as follows

$$\eta^{00} = -1 \ , \qquad \eta^{11} = \eta^{22} = \eta^{33} = +1 \ , \qquad \eta^{ab} = 0 \quad \text{for} \quad a \neq b \ . \tag{3}$$

In Minkowski space-time, Dirac equation is usually written as($\hbar = c = 1$)[6]

$$\left(i\gamma^{a}\frac{\partial}{\partial x^{a}} - m \right)\psi = 0 \ . \qquad a = 0, 1, 2, 3. \tag{4}$$

Where γ's are Dirac matrices, which satisfy

$$\gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a} = -2\eta^{ab} \ . \tag{5}$$

[†]In this letter, using Roman suffixes to refer to the bases of local Minkowski frame; using Greek suffixes to refer to curvilinear coordinates of space-time.

In my former paper[3], I had given the concept of daor field, which can be regarded as the square root of space-time metric. Daor field $h^a{}_\mu$ or $H_a{}^\mu$ satisfies

$$g_{\mu\nu} = h^{*}_{\mu}{}^{a}\eta_{ab}h^{b}{}_{\nu} \ , \qquad G^{\mu\nu} = H^{*\mu}{}_{a}\eta^{ab}H_{b}{}^{\nu} \ , \qquad g_{\mu\nu}G^{\nu\lambda} = G^{\lambda\nu}g_{\nu\mu} = \delta^{\lambda}_{\mu} \ . \tag{6}$$

where $*$ denotes complex conjugation. Set $h^a = h^a{}_\mu \mathrm{d}x^\mu$, which is daor field 1-form. By defining the Hermitean conjugate of daor field $h^a{}_\mu$ or $H_a{}^\mu$ as follows

$$(h^a{}_\mu)^\dagger = h^{*}_{\mu}{}^{a} \ , \qquad (H_a{}^\mu)^\dagger = H^{*\mu}{}_a \ , \tag{7}$$

we can easily acquire the following relations

$$g = h^\dagger \eta h \ , \quad G = H^\dagger \eta H \ , \quad G = g^{-1} \ , \quad H^\dagger = h^{-1} \ , \quad H = (h^\dagger)^{-1} \ . \tag{8}$$

By using the same definition of inner product as in differential geometry, the inner product of the vector $U = U^\mu \frac{\partial}{\partial x^\mu}$ and the covector $v = v_\nu \mathrm{d}x^\nu$ can be expressed as follows

$$< U, v > = < v, U > = U^\mu v_\mu = v_\mu U^\mu = U^{*a} v_a = v^*_a U^a \ . \tag{9}$$

Where U^{*a}, U^a, v^*_a and v_a are given by

$$U^{*a} = U^\mu h_\mu^{*a} \ , \quad U^a = h^a{}_\mu U^\mu \ , \quad v^*_a = v_\mu H^{*\mu}{}_a \ , \quad v_a = H_a{}^\mu v_\mu \ . \tag{10}$$

For simplicity, I will do not distinguish between the component U^a, v_a and its complex conjugate because they all can be transferred into real component in curvilinear coordinates x^μ's by corresponding form of daor field. In this daor geometry, the exterior derivative and exterior product have the same definitions and properties as in ordinary real vierbein geometry.

Now let us discuss a kind of gauge groups, which are the subgroups of U(1,3) group, and the element of gauge group can be written as

$$S^a{}_b(x) = e^{iZ^a{}_b(x)} \ . \tag{11}$$

Here, $Z^a{}_b$ is 4×4 matrix, which is traceless and Hermitean, of course, should be the function of curvilinear coordinates. So $S^a{}_b$ is the 4-dimensional representation of gauge group. Some groups such as U(1), SU(2) and SU(3) all satisfy the condition (11).

An intrinsic rotation of daor field is

$$h^a \to h'^a = S^a_{\ b} h^b \ , \tag{12}$$

here $S^a_{\ b}$ satisfies

$$S^{*c}_a \eta_{cd} S^d_{\ b} = \eta_{ab} \ . \tag{13}$$

From references[3, 1], it is known that under the intrinsic rotation of daor field the complex affine connection 1-form ω^a_b transforms as follows

$$\omega'^a_{\ b} = S^a_{\ c} \omega^c_{\ d} (S^{-1})^d_{\ b} + S^a_{\ c} (\mathrm{d}S^{-1})^c_{\ b} \ , \tag{14}$$

Covariance of the daor field equations under local gauge group directly leads to the introduction of Yang-Mills gauge fields[7]. Similarly, I separate gauge field $B^a_{\ b\mu}$ from the complex affine connection $\omega^a_{\ b\mu}$, say, write $\omega^a_{\ b}$ as

$$\omega^a_{\ b} = \Omega^a_{\ b} + i\epsilon' B^a_{\ b} \ , \tag{15}$$

where ϵ' is the coupling constant of gauge field, $B^a_{\ b}$ is also 1-form. Under the gauge rotation of daor field (12), $B^a_{\ b}$ transforms as follows

$$B'^a_{\ b} = S^a_{\ c} B^c_{\ d} (S^{-1})^d_{\ b} + \frac{1}{\epsilon'} \ S^a_{\ c} (\mathrm{d}S^{-1})^c_{\ b} = S^a_{\ c} B^c_{\ d} (S^{-1})^d_{\ b} + \frac{1}{\epsilon'} \ \mathrm{d}Z^c_{\ b} \ , \tag{16}$$

and $\Omega^a_{\ b}$ satisfies

$$\Omega'^a_{\ b} = S^a_{\ c} \Omega^c_{\ d} (S^{-1})^d_{\ b} \ . \tag{17}$$

As having been given in the paper of Yang and Mills[7], the gauge field strengths corresponding to gauge field $B^a_{\ b}$ is given by‡

$$F^a_{\ b} = \mathrm{d}B^a_{\ b} + i\epsilon' B^a_{\ c} \wedge B^c_{\ b} \ . \tag{18}$$

Where $F^a_{\ b}$ is a 2-form, which is a part of the curvature 2-form defined by

$$R^a_{\ b} = \mathrm{d}\omega^a_{\ b} + \omega^a_{\ c} \wedge \omega^c_{\ b} \ . \tag{19}$$

‡In this letter, d and $\wedge$ are exterior derivative and exterior product operators respectively.

It is stressed that F^a_b has quite the same symmetric characters as $R^a{}_b$. It is well known that the stress-energy tensor for this gauge field can be written as

$$4\pi T_{\mu\nu} = \mathrm{tr}(F_\mu{}^\alpha F_{\nu\alpha}) - \frac{1}{2}\, g_{\mu\nu}\mathrm{tr}(F_{\alpha\beta}F^{\alpha\beta}) \ , \tag{20}$$

Here 'tr' means operation of acquiring the trace of a matrix. In the following, I will give the coupling equation of daor field with gauge fields when the stress-energy tensor in Eq.(1) is the form of Eq.(20).

Yeah, I found that the doar-gauge field coupling equation can be written as follows

$$\mathrm{d}h^a + (\omega^a{}_b + i\ \epsilon\ \gamma^c\ F^a_{\ bc}) \wedge\ h^b = 0\ . \tag{21}$$

Where ϵ is the coupling constant, which equal to the square root of the newtonian gravitational constant, namely $\epsilon = \sqrt{G}$. It should be noted that $\gamma^c F^a_{\ bc}$ is a 1-form also, what is to say

$$\gamma^c F^a_{\ bc\mu} dx^\mu = \gamma^\mu F^a_{\ b\mu\nu} dx^\nu\ . \tag{22}$$

Eq.(21) also demonstrates that complex daor field should also have spinor suffixes.

I will prove that Einstein's gravitational equation can be deduced from Eq.(21). Set $\epsilon = 1$ in the process of proving for simplicity. Firstly, define the operator 1-form

$$\hat{W} \equiv \delta^a{}_b \mathrm{d} + (\omega^a{}_b + i\gamma^c F^a_{\ bc}) \wedge\ , \tag{23}$$

then, Eq.(21) becomes $\hat{W}h = 0$. Multiplying both sides of Eq.(21) by operator $\hat{W}$, we acquire

$$\begin{aligned} 0 &= \hat{W}\hat{W}h \\ &= [\mathrm{d}(\omega^a{}_b + i\gamma^c F^a_{\ bc}) + (\omega^a{}_e + i\gamma^c F^a_{\ ec}) \wedge (\omega^e{}_b + i\gamma^c F^e_{\ bc})] \wedge h^b \\ &= \left[R^a{}_b + i\gamma^c(\mathrm{d}F^a_{\ bc} + \omega^a{}_e \wedge F^e_{\ bc} + F^a_{\ ec} \wedge \omega^e{}_b) - \gamma^c\gamma^d F^a_{\ ec} \wedge F^e_{\ db}\right] \wedge h^b\ . \end{aligned} \tag{24}$$

The covariant derivative of a differential form V^a_b of degree p is defined as[1]

$$DV^a_b = \mathrm{d}V^a_b + \omega^a{}_c \wedge V^c_b - (-1)^p V^a_c \wedge \omega^c{}_b\ . \tag{25}$$

Because F^a_b is the gauge field strength, it satisfies the following Bianchi identities:

$$DF^a_{\ b} = 0 \ . \tag{26}$$

As I have stressed that $\gamma^c F^a_{bc}$ is a 1-form, Eq.(25) and Eq.(26) then make sure that $\gamma^c(\mathrm{d}F^a_{\ bc} + \omega^a_{\ e} \wedge F^e_{\ bc} + F^a_{\ ec} \wedge \omega^e_{\ b}) = 0$. So Eq.(24) becomes

$$R^a_{\ b} = \gamma^c \gamma^d F^a_{\ ec} \wedge F^e_{\ db} \ . \tag{27}$$

Transferring $R^a_{\ b}$ into the curvilinear coordinates of space-time, from Eq.(27) we obtain

$$R^\alpha_{\ \beta\mu\nu} = -\gamma^c\gamma^d(F^\alpha_{\ ec\nu}F^e_{\ \beta d\mu} - F^e_{\ \beta d\mu}F^\alpha_{\ ec\nu}) \tag{28}$$

Let us contract the suffixes α and μ in $R^\alpha_{\ \beta\mu\nu}$

$$\begin{aligned} R_{\beta\nu} &= -\sum_\alpha F^\alpha_{\ ec\nu}F^e_{\ \beta d\alpha}(\gamma^c\gamma^d + \gamma^d\gamma^c) \\ &= 2\sum_a F^a_{\ e\lambda\nu}F^{e\ \lambda}_{\ \beta\ \ a} \\ &= 2\mathrm{tr}(F_{\lambda\nu}F^\lambda_{\ \beta}) \ . \end{aligned} \tag{29}$$

Resuming the value of ϵ, we can acquire Einstein's gravitational equation (1) and the stress-energy tensor (20).

When there are different categories of gauge fields in the space-time, the coupling equation can be extended as follows

$$\mathrm{d}h^a + (\omega^a_{\ b} + i\ \epsilon\ \gamma^c\ F^a_{\ bc}) \wedge\ h^b = 0 \ , \qquad F^a_{\ b\mu\nu} = \sum_\tau {}^\tau F^a_{\ b\mu\nu} \ , \tag{30}$$

where ${}^\tau F^a_{\ b\mu\nu}$ denotes the strength of different gauge fields.

The coupling equation (21) demonstrates that the coupling constant ϵ is unrelated with the category of gauge field. This reflects the generality of gravitation. I believe, the coupling constant between daor field and spinor field should also be ϵ. The coupling between daor field and spinor field will be discussed in forthcoming papers.

The coupling equation (21) also indicates that only daor field can express the intrinsic harmony of different fields.

Conclusion: The general form of gauge fields are discussed. All gauge fields originate the invariance of local intrinsic rotation of doar field. The coupling equation is submitted, from which Einstein's equation can be obtained.

Acknowledgement: I would like to thank my parents , Long-Mei Zhu and You-Ping Dai for their help and encouragement.

References

[1] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. **66**, 213 (1980).

[2] T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).

[3] X. B. Huang, *"New Geometric Formalism for Gravity Equation in Empty Space"*, hep-th/0402139.

[4] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, San Francisco, 1973).

[5] P. A. M. Dirac, *General theory of relativity* (John Wiley & sons, Inc. , 1975).

[6] P. A. M. Dirac, Max-Planck-Festschrift, 339 (1958).

[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).